

FINITENESS PROPERTIES OF THE JOHNSON SUBGROUPS

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ABSTRACT. The main goal of this note is to provide evidence that the first rational homology of the Johnson subgroup $K_{g,1}$ of the mapping class group of a genus g surface with one marked point is finite-dimensional. Building on work of Dimca-Papadima [4], we use symplectic representation theory to show that, for all $g > 3$, the completion of $H_1(K_{g,1}, \mathbb{Q})$ with respect to the augmentation ideal in the rational group algebra of \mathbb{Z}^{2g} is finite-dimensional. We also show that the terms of the Johnson filtration of the mapping class group have infinite-dimensional rational homology in some degrees in almost all genera, generalizing a result of Akita.

1. INTRODUCTION

Let π denote the fundamental group of a closed orientable surface S_g of genus $g \geq 2$ and let $\Gamma_{g,r}$ denote the mapping class group of S_g with $r \leq 1$ marked points. The n th Johnson subgroup $K_{g,1}(n)$ of $\Gamma_{g,1}$ and the n th outer Johnson subgroup $K_g(n)$ of $\Gamma_g := \Gamma_{g,0}$ are defined, respectively, to be the kernels of the natural maps

$$\Gamma_{g,1} \rightarrow \text{Aut}(\pi/\pi^{(n+1)}) \quad \Gamma_g \rightarrow \text{Out}(\pi/\pi^{(n+1)}).$$

Here $\pi^{(n)}$ denotes the n th term of the lower central series of π . The classical Torelli groups $T_{g,1}$ and T_g are recovered by taking $n = 1$, and the original Johnson subgroups $K_{g,1}$ and K_g are recovered by taking $n = 2$.

The finiteness properties of the Johnson subgroups are poorly understood. For example, it is not known whether these are finitely generated for even a single $n \geq 2$. In [4], Dimca-Papadima showed that $H_1(K_g, \mathbb{C})$ is a finite-dimensional vector space as long as $g \geq 4$. It is currently unknown whether $H_1(K_3, \mathbb{Q})$ is finite-dimensional. On the other hand, in [1] Akita showed that $H_\bullet(K_g, \mathbb{Q})$ and $H_\bullet(K_{g,1}, \mathbb{Q})$ are infinite-dimensional when $g \geq 7$, so K_g and $K_{g,1}$ must have some infinite-rank homology in these cases. Mess has shown in [13] that $K_2 = T_2$ is a free group of countably infinite rank, implying that $H_1(K_2, \mathbb{Q})$ is infinite-dimensional. It is currently unknown whether $H_1(K_{g,1}, \mathbb{Q})$ is finite-dimensional when $g \geq 3$.

The main goal of this note is to provide evidence that $H_1(K_{g,1}, \mathbb{Q})$ is finite-dimensional. Recall that if a finite-dimensional k -vector space V is also a module over a commutative noetherian k -algebra A , then the completion of V with respect to any ideal $I \subset A$ is a finite-dimensional k -vector space. Let $H = \pi^{ab}$. Then $H_1(K_{g,1}, \mathbb{Q})$ is a $\mathbb{Q}H$ -module¹. Let J denote the augmentation ideal of $\mathbb{Q}H$.

Theorem 1.1. *For each $g \geq 4$ the J -adic completion $H_1(K_{g,1}, \mathbb{Q})^\wedge$ is finite-dimensional.*

Theorem 1.1 does not automatically imply that $H_1(K_{g,1}, \mathbb{Q})$ is finite-dimensional, though it is consistent with that hypothesis.

¹This will be shown in Section 4.

We now provide an overview of the paper. In Section 2 we recall the basics of Torelli groups and Johnson subgroups, including the higher Johnson homomorphisms defined by Morita [14]. In Section 3 we review a result of Papadima-Suciu [16] that will provide us with a Birman-type exact sequence for the Johnson subgroups that will be used throughout the paper.

Section 4 is dedicated to the structure of $H_1(\pi^{(2)}, \mathbb{Q})$ and its $K_{g,1}$ -coinvariants. The main result of this section is that the J -adic filtration on $H_1(\pi^{(2)}, \mathbb{Q})_{K_{g,1}}$ stabilizes. This is the main technical result needed for the proof of Theorem 1.1. Along the way, we will study the graded quotients of $H_1(\pi^{(2)}, \mathbb{Q})$ associated with the J -adic filtration, each of which has a natural $\mathrm{Sp}_g(\mathbb{Z})$ -module structure. Following the conventions of [8], let $V(\lambda)$ denote the irreducible representation of $\mathrm{Sp}_g(\mathbb{Q})$ with highest weight λ .

Proposition 1. *For each $n \geq 0$ there is a natural $\mathrm{Sp}_g(\mathbb{Z})$ -equivariant isomorphism*

$$V(n\lambda_1 + \lambda_2) \longrightarrow \frac{J^n H_1(\pi^{(2)}, \mathbb{Q})}{J^{n+1} H_1(\pi^{(2)}, \mathbb{Q})}$$

Section 5 is of a slightly different flavor. Here we investigate the infinite-dimensional homology of the Johnson subgroups and show that, in sufficiently large genus, the higher Johnson subgroups have infinite-dimensional homology in some degrees.

Proposition 2. *For each $n \geq 2$ and $g \geq 7$, the homology spaces $H_\bullet(K_{g,1}(n), \mathbb{Q})$ and $H_\bullet(K_g(n), \mathbb{Q})$ are infinite-dimensional.*

This is a direct generalization of Akita's result. We will conclude the paper by providing a condition under which Proposition 2 could be sharpened considerably.

Theorem 1.2. *If $H_1(\pi^{(2)}, \mathbb{Q})_{K_{g,1}}$ is infinite-dimensional then for each $n \geq 2$ the direct sum $H_2(K_g(n), \mathbb{Q}) \oplus H_1(K_{g,1}(n), \mathbb{Q})$ is infinite-dimensional.*

Corollary 1. *If $H_1(\pi^{(2)}, \mathbb{Q})_{K_{g,1}}$ is infinite-dimensional, then for each $n \geq 2$ either $K_g(n)$ is not finitely presented or $K_{g,1}(n)$ is not finitely generated.*

2. PRELIMINARIES

2.1. Mapping Class Groups and Torelli Groups. Let $S_{g,n}$ denote a closed orientable surface of genus g with n marked points, where $n \leq 1$. The *mapping class group* $\Gamma_{g,n}$ is the group of isotopy classes of orientation preserving diffeomorphisms of S_g that fix the marked point. The *Torelli group* $T_{g,n}$ is the kernel of the natural representation $\Gamma_{g,n} \rightarrow \mathrm{Sp}(H) \cong \mathrm{Sp}_g(\mathbb{Z})$, where $H = H_1(S_g, \mathbb{Z})$. We will omit the decoration n when it is equal to 0.

Johnson proved in [10] that T_g is generated by finitely many bounding pair maps when $g \geq 3$, and this implies that $T_{g,n}$ is finitely generated for all pairs (g, n) as long as $g \geq 3$. Mess [13] proved that T_2 is a free group of countably infinite rank, and showed that T_2 is freely generated by a set of Dehn twists on separating curves. It is currently unknown whether T_g is finitely presented for even a single $g \geq 3$.

2.2. Johnson Subgroups. In [14], Morita introduced a family of $\Gamma_{g,1}$ -equivariant homomorphisms

$$\tau_{g,1}(n) : K_{g,1}(n) \rightarrow \mathrm{Hom}(H, \mathcal{L}_{n+1}(\pi))$$

defined by

$$(1) \quad \varphi \rightarrow \{\bar{x} \rightarrow \overline{\varphi(x)x^{-1}}\},$$

where $\mathcal{L}_k(\pi) = \pi^{(k)}/\pi^{(k+1)}$. The homomorphism $\tau_{g,1}(n)$ is the n th Johnson homomorphism.

Because the graded Lie algebra associated to the lower central series of π is center-free (see Proposition 3 below), there is a natural $\mathrm{Sp}_g(\mathbb{Z})$ -equivariant inclusion $\mathcal{L}_n(\pi) \hookrightarrow \mathrm{Hom}(H, \mathcal{L}_{n+1}(\pi))$. The n th outer Johnson homomorphism $\tau_g(n)$ is a map

$$\tau_g(n) : K_g(n) \rightarrow \mathrm{Hom}(H, \mathcal{L}_{n+1}(\pi))/\mathcal{L}_n(\pi)$$

defined by a formula analogous to (1) above. It can be shown that the Johnson homomorphisms fit into exact sequences

$$1 \longrightarrow K_{g,1}(n+1) \longrightarrow K_{g,1}(n) \xrightarrow{\tau_{g,1}(n)} \mathrm{Hom}(H, \mathcal{L}_{n+1}(\pi))$$

$$1 \longrightarrow K_g(n+1) \longrightarrow K_g(n) \xrightarrow{\tau_g(n)} \mathrm{Hom}(H, \mathcal{L}_{n+1}(\pi))/\mathcal{L}_n(\pi).$$

The classical Johnson homomorphism $\tau_{g,1}(1)$ was originally defined by Johnson in [10]. Johnson showed that the image of $\tau_{g,1}(1)$ is isomorphic to $\Lambda^3 H$ and that the image of τ_g is isomorphic to $\Lambda^3 H/\theta \wedge H$, where $\theta \in \Lambda^2 H$ is the symplectic form. It is, in general, very difficult to compute the image of the higher Johnson homomorphisms². The kernels $K_{g,1}$ and K_g of $\tau_{g,1}(1)$ and $\tau_g(1)$, respectively, are the classical Johnson subgroups. Little is known about the structure of $K_{g,1}$ and K_g , aside from a handful of basic results. In [11], Johnson proved the following beautiful theorem.

Theorem 2.1 (Johnson). *For all $g \geq 2$, the Johnson subgroup K_g is generated by Dehn twists on separating simple closed curves.*

It is a difficult open problem to determine whether or not K_g is finitely generated and, in light of [4], this cannot be determined by considering the dimension of $H_1(K_g, \mathbb{Q})$ alone, as long as $g \geq 4$.

3. BIRMAN SEQUENCE FOR THE JOHNSON SUBGROUPS

One of the key results needed throughout the paper is a Birman-type exact sequence that relates the n th Johnson subgroup to the n th outer Johnson subgroup. There is a forgetful map $T_{g,1} \rightarrow T_g$ obtained by forgetting the marked point. This is a surjective homomorphism. The kernel can be identified with π via the push map $\mathcal{P} : \pi \rightarrow T_{g,1}$ which sends an element $x \in \pi$ to the isotopy class of the diffeomorphism of $S_{g,1}$ obtained by dragging the marked point along x . For any $\gamma \in \pi$, the push map satisfies $\mathcal{P}(x)(\gamma) = x\gamma x^{-1}$.

Theorem 3.1 (Birman). *There is an exact sequence*

$$1 \rightarrow \pi \xrightarrow{\mathcal{P}} T_{g,1} \rightarrow T_g \rightarrow 1.$$

²Hain [8], Morita [15] and others have determined $\mathrm{Im}(\tau_{g,1}(n)) \otimes \mathbb{Q}$ up to $n = 6$

We now review some of the results from [16]. Let G be a group and let $G^{(n)}$ denote the n th term of the lower central series of G . The Torelli group \mathcal{T}_G of a group G is defined to be the kernel of the natural map $\text{Aut}(G) \rightarrow \text{Aut}(G/G^{(2)})$. The outer Torelli group $\widetilde{\mathcal{T}}_G$ is defined to be the kernel of the map $\text{Out}(G) \rightarrow \text{Out}(G/G^{(2)})$. The Johnson-Andreadakis filtration $F^s \mathcal{T}_G$ of \mathcal{T}_G is defined by

$$F^s \mathcal{T}_G = \ker \left(\text{Aut}(G) \rightarrow \text{Aut}(G/G^{(s+1)}) \right).$$

The Johnson-Andreadakis filtration $F^s \widetilde{\mathcal{T}}_G$ of $\widetilde{\mathcal{T}}_G$ is defined completely analogously.

In [16] Papadima-Suciu have worked out the relation between the terms of the filtrations on \mathcal{T}_G and $\widetilde{\mathcal{T}}_G$, at least in certain cases.

Recall that a group G is said to be residually nilpotent if $\bigcap_{n=1}^{\infty} G^{(n)} = 1$.

Theorem 3.2 (Papadima-Suciu, [16]). *Suppose that G is a residually nilpotent group whose associated graded Lie algebra $\text{gr } G$ has trivial center. Then for each $n \geq 1$ there is an exact sequence*

$$(2) \quad 1 \rightarrow G^{(n)} \rightarrow F^n \mathcal{T}_G \rightarrow F^n \widetilde{\mathcal{T}}_G \rightarrow 1.$$

The next proposition consists of standard facts which are recorded here for the reader's convenience.

Proposition 3. *The fundamental group π of a closed surface of genus g is residually nilpotent, and the associated graded Lie algebra associated to its lower central series has trivial center.*

Proof. The first claim is a well-known result from combinatorial group theory and can be found in [3]. The second claim follows from the fact, proven in [2], that the associated graded Lie algebra $\text{gr}^{\text{LCS}} \mathfrak{p}$ of the Malcev Lie algebra \mathfrak{p} of π has trivial center. This, combined with the fact the lower central quotients of π are torsion-free [12] and that $\text{gr}^{\text{LCS}} \mathfrak{p} \cong \text{gr } \pi \otimes \mathbb{R}$ [8], implies that $\text{gr } \pi$ has trivial center. \square

The extended mapping class group $\Gamma_{g,1}^{\pm}$ is defined to be the group of isotopy classes of all diffeomorphisms of S_g that fix the marked point (not just the orientation-preserving ones). The extended mapping class group Γ_g^{\pm} is defined completely analogously.

Theorem 3.3 (Dehn-Nielsen-Baer, c.f. [6]). *The natural maps*

$$\Gamma_{g,1}^{\pm} \rightarrow \text{Aut}(\pi) \quad \Gamma_g^{\pm} \rightarrow \text{Out}(\pi)$$

are isomorphisms.

Theorem 3.3 implies that the Torelli group and outer Torelli group of π coincide with $T_{g,1}$ and T_g , respectively. Combined with Theorem 3.2 we easily obtain the following.

Corollary 2. *For each $n \geq 1$ there is an exact sequence*

$$1 \rightarrow \pi^{(n)} \xrightarrow{\mathcal{P}} K_{g,1}(n) \rightarrow K_g(n) \rightarrow 1$$

It is easily checked that the extension in Corollary 2 is π -equivariant with respect to the conjugation action on $\pi^{(n)}$ and $K_{g,1}(n)$ and the trivial action on $K_g(n)$.

4. THE STRUCTURE OF THE $K_{g,1}$ -COINVARIANTS

In this section we determine the $\mathrm{Sp}_g(\mathbb{Z})$ -module structure on the graded quotients of $H_1(\pi^{(2)}, \mathbb{Q})$. The key point is that each graded quotient is irreducible. We prove this assertion by explicit computation. This fact leads to the main technical result (Corollary 5 below) concerning the space $H_1(\pi^{(2)}, \mathbb{Q})_{K_{g,1}}$ of $K_{g,1}$ -coinvariants needed for the proof of Theorem 1.1.

4.1. The Alexander Invariant as a Module. Let G be a group. The abelian group $H_1(G^{(2)}, \mathbb{Z})$ is the Alexander invariant of G . It is naturally a $\mathbb{Z}G^{ab}$ module via the conjugation action of G . When G is finitely generated, the $\mathbb{Z}G^{ab}$ -module $H_1(G^{(2)}, \mathbb{Z})$ is also finitely generated. When G is finitely presented and G^{ab} is torsion-free, Fox differential calculus can be used to obtain a presentation for $H_1(G^{(2)}, \mathbb{Z})$ as a $\mathbb{Z}H$ -module [17].

Let $F = \langle x_1, \dots, x_r \rangle$ be a free group of rank r , and let $\phi : F \rightarrow H$ denote its abelianization. The Fox derivatives D_j , $j = 1, \dots, r$ are abelian group homomorphisms $\mathbb{Z}F \rightarrow \mathbb{Z}H$ satisfying the following properties:

- (1) $D_i(x_j) = \delta_{ij} \cdot 1$
- (2) $D_i(xy) = D_i(x) + \phi(x)D_i(y) \quad x, y \in F$.

Suppose that $\langle F | s_1, \dots, s_k \rangle$ is a finite presentation for a group G , and assume that the abelianization of G is a free abelian group H . Then $H_1(G^{(2)}, \mathbb{Z})$ can be realized as a $\mathbb{Z}H$ -submodule of the quotient of $(\mathbb{Z}H)^{\oplus r}$ whose presentation matrix is $(D_j(s_i))$. We use the presentation $\langle x_1, \dots, x_{2g} | [x_1, x_{g+1}] \cdots [x_g, x_{2g}] \rangle$ of π in order to describe $H_1(\pi^{(2)}, \mathbb{Z})$. Define $\vartheta = [x_1, x_{g+1}] \cdots [x_g, x_{2g}]$.

Lemma 4.1. *The matrix $(D_j(\vartheta))$ is equal to the row vector*

$$(1 - x_{g+1}, \dots, 1 - x_{2g}, -(1 - x_1), \dots, -(1 - x_g)).$$

Let $e_j \in \mathbb{Z}H^{\oplus 2g}$ denote the column vector whose j th entry is 1 and whose other entries are 0. Define $\Theta = \sum_{j=1}^g (1 - x_{g+j})e_j - (1 - x_j)e_{g+j} \in (\mathbb{Z}H)^{\oplus 2g}$. Then $H_1(\pi^{(2)}, \mathbb{Z})$ is naturally a $\mathbb{Z}H$ -submodule of $(\mathbb{Z}H)^{\oplus 2g}/(\Theta)$, where (Θ) denotes the submodule of $(\mathbb{Z}H)^{\oplus 2g}$ spanned by Θ .

Lemma 4.2. *The $\mathbb{Z}H$ -module $(\mathbb{Z}H)^{\oplus 2g}/(\Theta)$ is torsion-free.*

Proof. Suppose that $x = \sum_{j=1}^{2g} a_j e_j$ and that the image of x in $(\mathbb{Z}H)^{\oplus 2g}/(\Theta)$ is a torsion element. Then there exist non-zero $P, Q \in \mathbb{Z}H$ such that

$$P \sum_{j=1}^g a_j e_j = Q \sum_{j=1}^g (1 - x_{j'})e_j - (1 - x_j)e_{j'}$$

where $j' = g + j$. This gives a system of equations

$$(3) \quad Pa_j = Q(1 - x_{j'}) \quad Pa_{j'} = -Q(1 - x_j)$$

for each j . Multiplying the first by $a_{j'}$ and the second by a_j leads to the equation

$$Q(a_{j'}(1 - x_{j'}) + a_j(1 - x_j)) = 0.$$

Since $(1 - x_j)$ and $(1 - x_{j'})$ are prime ideals in the integral domain $\mathbb{Z}H$, we have $a_j \in (1 - x_{j'})$ and $a_{j'} \in (1 - x_j)$. Now write $a_j = r_j(1 - x_{j'})$ and $a_{j'} = r_{j'}(1 - x_j)$. Then by (3) we obtain the equations

$$(4) \quad (Pr_j - Q)(1 - x_{j'}) = 0 \quad (Pr_{j'} + Q)(1 - x_j) = 0$$

implying that $P(r_j + r_{j'}) = 0$. That is, $r_j = -r_{j'}$. From (4) we also deduce that r_j does not depend on j . That is, r_j assumes a single value $r \in \mathbb{Z}H$. Finally, we are able to write

$$x = r \sum_{j=1}^g (1 - x_{j'}) e_j - (1 - x_j) e_{j'} = r\Theta.$$

This completes the proof. \square

Corollary 3. *The Alexander invariant of π is a torsion-free $\mathbb{Z}H$ -module.*

The rational Alexander invariant $A := H_1(\pi^{(2)}, \mathbb{Q})$ is a $\mathbb{Q}H$ -module. The following is deduced at once from Corollary 3.

Corollary 4. *The rational Alexander invariant A is a torsion-free $\mathbb{Q}H$ -module.*

Let J denote the augmentation ideal in $\mathbb{Q}H$ and let $\text{gr}_\bullet A$ denote the graded \mathbb{Q} -vector space associated to the J -adic filtration on A .

Proposition 4. *For each $k \geq 0$, the graded quotients $\text{gr}_k A$ are non-zero.*

Proof. Because π is finitely generated, A is a finitely generated module over $\mathbb{Q}H$. Since $\mathbb{Q}H$ is noetherian, the Krull intersection theorem applies, (see [5], p.152) and because A is a torsion-free $\mathbb{Q}H$ module, the intersection

$$\bigcap_{n=1}^{\infty} J^n A$$

must vanish. This implies that the J -adic filtration does not stabilize and therefore that the graded quotients are all non-zero. \square

4.2. Computing the Graded Quotients. Following the conventions of [8], denote the irreducible representation of $\text{Sp}_g(\mathbb{Q})$ with highest weight λ by $V(\lambda)$. Define $H_{\mathbb{Q}} = H \otimes \mathbb{Q}$.

Proposition 5. *There is a surjective $\text{Sp}_g(\mathbb{Z})$ -equivariant map*

$$\varphi : \text{Sym}^n(H_{\mathbb{Q}}) \otimes \Lambda^2(H_{\mathbb{Q}}) \rightarrow \text{gr}_n A$$

that satisfies

$$(5) \quad \varphi(x_1^{n_1} \cdots x_{2g}^{n_{2g}} \otimes y \wedge z) = (x_1 - 1)^{n_1} \cdots (x_{2g} - 1)^{n_{2g}} [y, z].$$

Proof. There is a surjective map

$$J^n / J^{n+1} \otimes_{\mathbb{Q}} A_H \rightarrow \text{gr}_n A$$

given by multiplication which is easily seen to be $\Gamma_{g,1}$ -equivariant (where $\Gamma_{g,1}$ acts diagonally on the left). Since the Torelli group $T_{g,1}$ acts trivially on both sides, this map is actually $\text{Sp}_g(\mathbb{Z})$ -equivariant. The 5-term exact sequence for the extension $\pi^{(2)} \rightarrow \pi \rightarrow H$ produces an $\text{Sp}_g(\mathbb{Z})$ -equivariant isomorphism $A_H \rightarrow \Lambda_0^2 H$ defined by $[y, z] \rightarrow y \wedge z$. There is also an $\text{Sp}_g(\mathbb{Z})$ -equivariant isomorphism $\text{Sym}^n(H_{\mathbb{Q}}) \rightarrow J^n / J^{n+1}$ defined by sending the degree n monomial $x_1^{n_1} \cdots x_{2g}^{n_{2g}} \rightarrow (x_1 - 1)^{n_1} \cdots (x_{2g} - 1)^{n_{2g}}$. Since $\Lambda^2 H_{\mathbb{Q}}$ decomposes as $\Lambda_0^2 H_{\mathbb{Q}} + \mathbb{Q}\theta$, and the class of θ vanishes in $H_1(\pi^{(2)}, \mathbb{Q})$, we obtain a well-defined map

$$\text{Sym}^n(H_{\mathbb{Q}}) \otimes \Lambda^2 H_{\mathbb{Q}} \rightarrow \text{gr}_{n+2} A$$

with the desired properties. \square

Let $\text{Sym}^\bullet(H_{\mathbb{Q}})$ denote the graded ring $\bigoplus_{n \geq 0} \text{Sym}^n(H_{\mathbb{Q}})$. The tensor product

$$\text{Sym}^\bullet(H_{\mathbb{Q}}) \otimes \Lambda^2 H_{\mathbb{Q}} = \bigoplus_{n \geq 0} \text{Sym}^n(H_{\mathbb{Q}}) \otimes \Lambda^2 H_{\mathbb{Q}}$$

is then both a graded $\text{Sym}^\bullet(H_{\mathbb{Q}})$ -module and a $\text{Sp}_g(\mathbb{Z})$ -module, and there is a natural surjective $\text{Sp}_g(\mathbb{Z})$ -equivariant map

$$\text{Sym}^\bullet(H_{\mathbb{Q}}) \otimes \Lambda^2 H_{\mathbb{Q}} \rightarrow \text{gr}_\bullet A$$

of graded $\text{Sym}^\bullet(H_{\mathbb{Q}})$ -modules.

Proposition 6. *When $n \geq 2$ the highest weight decomposition of $\text{Sym}^n(H_{\mathbb{Q}}) \otimes \Lambda^2 H$ is*

$$\begin{cases} V(n\lambda_1 + \lambda_2) + 2V(n\lambda_1) + V((n-2)\lambda_1 + \lambda_2) & g = 2 \\ V(n\lambda_1 + \lambda_2) + 2V(n\lambda_1) + V((n-1)\lambda_1 + \lambda_3) + V((n-2)\lambda_1 + \lambda_2) & g \geq 3 \end{cases}$$

The highest weight decomposition of $H_{\mathbb{Q}} \otimes \Lambda^2 H$ is

$$\begin{cases} V(\lambda_1 + \lambda_2) + 2V(\lambda_1) & g = 2 \\ V(\lambda_1 + \lambda_2) + 2V(\lambda_1) + V(\lambda_3) & g \geq 3 \end{cases}$$

Proof. Let $i : \Lambda^3 H \rightarrow H \otimes \Lambda^2 H$ be the $\text{Sp}_g(\mathbb{Q})$ -equivariant map given by

$$i(x \wedge y \wedge z) = x \otimes y \wedge z + y \otimes z \wedge x + z \otimes x \wedge y.$$

When $g = 2, 3$ it is straightforward to show that the following vectors are highest weight vectors in $\text{Sym}^n(H_{\mathbb{Q}}) \otimes \Lambda^2 H_{\mathbb{Q}}$ when the formulae make sense:

Irreducible Summand	Highest Weight Vector(s)
$V(n\lambda_1 + \lambda_2)$	$a_1^n \otimes a_1 \wedge a_2$
$V(n\lambda_1)$	$a_1^n \otimes \theta \quad a_1^{n-1} \cdot i(a_1 \wedge \theta)$
$V((n-1)\lambda_1 + \lambda_3)$	$a_1^{n-1} \cdot i(a_1 \wedge a_2 \wedge a_3)$
$V((n-2)\lambda_1 + \lambda_2)$	$a_1^{n-2} a_2 \cdot i(a_1 \wedge \theta) - a_1^{n-1} \cdot i(a_2 \wedge \theta)$

TABLE 1. Highest weight vectors in $\text{Sym}^n(H_{\mathbb{Q}}) \otimes \Lambda^2 H_{\mathbb{Q}}$

This shows that $\text{Sym}^n(H_{\mathbb{Q}}) \otimes \Lambda^2 H_{\mathbb{Q}}$ contains the summands claimed in the statement of the proposition. From a dimension count, using, for example, the Weyl character formula, it is readily checked that the sum of the dimensions of these irreducible summands is equal to the dimension of $\text{Sym}^n(H_{\mathbb{Q}}) \otimes \Lambda^2 H_{\mathbb{Q}}$. Thus when $g = 2$ or 3 , we have the required irreducible decomposition. To handle the cases where $g \geq 4$ we apply the stability result on p.618 of [8], using the fact that $\Lambda^2 H_{\mathbb{Q}} = V(\lambda_2) + V(0)$. \square

We will need the following fact, which is easily proved using the Hall-Witt identity $[x, yz] = [x, y][x, z]^{y^{-1}}$, where $a^b = b^{-1}ab$.

Lemma 4.3. *The Jacobi identity holds in A , i.e. for any elements $x, y, z \in \pi$ we have that*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Proposition 7. *The image in $\text{gr}_n A$ of each highest weight vector in Table 1 except $a_1^n \otimes a_1 \wedge a_2$ vanishes.*

Proof. The vanishing of the image of each highest weight vector in Table 1 besides $a_1^n \otimes a_1 \wedge a_2$ follows readily from an application of the Jacobi Identity and the fact that θ maps to zero in A . On the other hand, $\text{gr}_n A \neq 0$, so the image of $a_1^n \otimes a_1 \wedge a_2$ must be non-zero. \square

We are now in a position to give the proof of Proposition 1. Recall that any finite-dimensional irreducible representation of $\text{Sp}_g(\mathbb{Q})$ is also an irreducible representation of $\text{Sp}_g(\mathbb{Z})$.

Proof of Proposition 1. For each $n \geq 0$, the $\text{Sp}_g(\mathbb{Z})$ -equivariant map

$$V(n\lambda_1 + \lambda_2) \rightarrow \text{gr}_n A$$

is surjective. Since $\text{gr}_n A \neq 0$ and $V(n\lambda_1 + \lambda_2)$ is an irreducible representation of $\text{Sp}_g(\mathbb{Z})$, this map must be an isomorphism by Schur's Lemma. \square

4.3. The Filtration Stabilizes. Let $J_{K_{g,1}}$ denote the augmentation ideal in the group algebra $\mathbb{Q}K_{g,1}$. Because $K_{g,1}$ is a normal subgroup of $\Gamma_{g,1}$, the subspace $J_{K_{g,1}}A$ of A is actually a $\mathbb{Q}H$ -submodule. It follows that space of coinvariants $A_{K_{g,1}}$ is naturally a $\mathbb{Q}H$ -module. Let $\text{gr}_\bullet A_{K_{g,1}}$ denote the graded \mathbb{Q} -vector space associated with the J -adic filtration on $A_{K_{g,1}}$.

For each $n \geq 0$, the canonical projection $A \rightarrow A_{K_{g,1}}$ induces a surjective $\text{Sp}_g(\mathbb{Z})$ -equivariant map

$$(6) \quad \psi_n : \text{gr}_n A \rightarrow \text{gr}_n A_{K_{g,1}}.$$

Since the domain is an irreducible $\text{Sp}_g(\mathbb{Z})$ -module, this map is either zero or an isomorphism.

To show that the J -adic filtration on $\text{gr}_N A_{K_{g,1}}$ must stabilize, we first exhibit a non-zero element of $\text{gr}_N A$ for some sufficiently large N . We then show that this particular element lies in the kernel of ψ_N . Since $\text{gr}_N A$ is irreducible, this implies that $\text{gr}_N A_{K_{g,1}} = 0$.

In the calculations that follow, we will use the generating set $\{a_j, b_j\}_{j=1}^{2g}$ for π pictured below in Figure 1 along with the separating curve c . Let T_c denote a Dehn twist on the separating simple closed curve c . Then $T_c \in K_{g,1}$.

Lemma 4.4. *The element $(a_1 - 1)(a_2 - 1)[a_1, b_1] \in A$ lies in $J_{K_{g,1}}A$.*

Proof. We have $T_c(a_1) = [[a_1, b_1], a_1]a_1$ and $T_c(a_2) = a_2$. By the Hall-Witt identity, then, the following identity is satisfied in $H_1(\pi^{(2)}, \mathbb{Q})$:

$$\begin{aligned} T_c[a_1, a_2] &= [[[a_1, b_1], a_1]a_1, a_2] \\ &= [a_1, a_2] + [[[a_1, b_1], a_1], a_2] \\ &= [a_1, a_2] + (a_1 - 1)(a_2 - 1)[a_1, b_1]. \end{aligned}$$

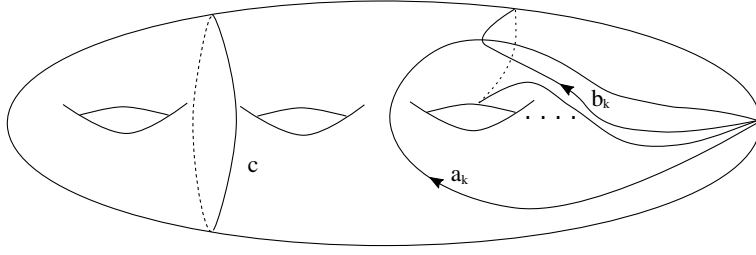
\square

From now on, we denote the element $(a_1 - 1)(a_2 - 1)[a_1, b_1]$ by w . The following property of w is crucial.

Lemma 4.5. *The vector $w \in A$ is non-zero.*

Proof. Observe first that $[a_1, b_1] \neq 0$, as it projects to the non-zero element $a_1 \wedge b_1$ via the canonical map

$$A \rightarrow A_H \cong \Lambda_0^2 H_{\mathbb{Q}},$$


 FIGURE 1. Generators for π and a separating curve.

where $\Lambda_0^2 H_{\mathbb{Q}}$ denotes the unique $\mathrm{Sp}_g(\mathbb{Q})$ -invariant complement of the trivial representation in $\Lambda^2 H_{\mathbb{Q}}$. Since A is a torsion-free $\mathbb{Q}H$ -module, we therefore also have $(a_1 - 1)(a_2 - 1)[a_1, b_1] \neq 0$. \square

Since A is a finitely generated torsion-free $\mathbb{Q}H$ -module, the Krull Intersection Theorem implies the following.

Lemma 4.6. *For each $g \geq 2$ there exists a unique integer $N_g \geq 2$ with the property that w is contained in $J^{N_g} A$ but not in $J^{N_g+1} A$.*

Let N denote the integer that is guaranteed to exist by Lemma 4.6. Then the class of w in $\mathrm{gr}_{N_g} A$ is non-zero. We claim that it lies in the kernel of the projection onto $\mathrm{gr}_{N_g} A_{K_{g,1}}$. To see this, observe that there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{p} & A_{K_{g,1}} \\ \uparrow & & \uparrow \\ J^{N_g} A & \xrightarrow{p'} & J^{N_g} A_{K_{g,1}} \end{array}$$

where the vertical maps are inclusions and the horizontal maps are the canonical projections. Because $w \in J_{K_{g,1}} A$ by Lemma 4.4, we have $p(w) = 0$. By commutativity of the diagram, we obtain $p'(w) = 0$. It follows that the class of w in $\mathrm{gr}_{N_g} A$ is mapped to 0 in $\mathrm{gr}_{N_g} A_{K_{g,1}}$.

Corollary 5. *The J -adic filtration of $A_{K_{g,1}}$ stabilizes.*

Proof. Since w does not vanish in $\mathrm{gr}_{N_g} A$, it follows from Schur's Lemma that ψ_{N_g} vanishes identically. This implies that $J^{N_g} A_{K_{g,1}} = J^k A_{K_{g,1}}$ for all $k \geq N_g$. \square

Corollary 6. *The J -adic completion $A_{K_{g,1}}^\wedge$ is isomorphic, as a $\mathbb{Q}H$ -module, to $A/J^{N_g} A$. In particular, it is a finite-dimensional vector space.*

Proof. This follows readily from Corollary 5 along with standard properties of completions. \square

We conclude this section with the proof of Theorem 1.1. By Corollary 2 there is an extension

$$(7) \quad 1 \rightarrow \pi^{(2)} \rightarrow K_{g,1} \rightarrow K_g \rightarrow 1$$

giving rise to a 5-term exact sequence with a segment

$$H_1(\pi^{(2)}, \mathbb{Q})_{K_{g,1}} \rightarrow H_1(K_{g,1}, \mathbb{Q}) \rightarrow H_1(K_g, \mathbb{Q}) \rightarrow 0.$$

Since (7) is π -equivariant, this is a sequence of $\mathbb{Q}H$ -modules. Note that H acts trivially on $H_1(K_g, \mathbb{Q})$.

Proof of Theorem 1.1. When $g \geq 4$, $H_1(K_g, \mathbb{Q})$ is a finite-dimensional vector space and therefore a finitely generated $\mathbb{Q}H$ -module. Since $H_1(\pi^{(2)}, \mathbb{Q})$ is a finitely generated $\mathbb{Q}H$ -module, so is $H_1(\pi^{(2)}, \mathbb{Q})_{K_{g,1}}$. The J -adic completion functor is exact on the category of finitely generated $\mathbb{Q}H$ -modules, so we have an exact sequence

$$\left(H_1(\pi^{(2)}, \mathbb{Q})_{K_{g,1}}\right)^\wedge \rightarrow H_1(K_{g,1}, \mathbb{Q})^\wedge \rightarrow H_1(K_g, \mathbb{Q}) \rightarrow 0.$$

Since $\left(H_1(\pi^{(2)}, \mathbb{Q})_{K_{g,1}}\right)^\wedge$ is finite-dimensional by Corollary 6, $H_1(K_{g,1}, \mathbb{Q})^\wedge$ must be finite-dimensional as well. \square

We conclude this section with a discussion of the structure of $J_{K_{g,1}}A$. Our first observation concerns how this subspace is situated relative to the J -adic filtration.

Lemma 4.7. *The submodule $J_{K_{g,1}}A$ is contained in J^2A .*

Proof. Suppose that $x_1, x_2 \in \pi$. By definition, any element $f \in K_{g,1}$ satisfies $f(x_j)x_j^{-1} \in \pi^{(3)}$. Let $\delta_j \in \pi^{(3)}$ be elements with the property that $f(x_j) = \delta_j x_j$. Then by the Hall-Witt identity we obtain

$$f([x_1, x_2]) = [\delta_1 x_1, \delta_2 x_2] = [x_1, \delta_2]^{\delta_1^{-1}} [x_1, x_2]^{\delta_2^{-1} \delta_1^{-1}} [\delta_1, \delta_2] [\delta_1, x_2]^{\delta_2^{-1}}.$$

This implies that the element $f - 1 \in J_{K_{g,1}}$ satisfies

$$(8) \quad (f - 1)[x, y] = (x_1 - 1) \cdot \delta_2 - (x_2 - 1) \cdot \delta_1 \in H_1(\pi^{(2)}, \mathbb{Q}).$$

Note that the image of the natural map $\pi^{(3)} \rightarrow A$ is contained in the subspace JA . This observation, combined with (8), implies that each $f \in K_{g,1}$ satisfies $(f - 1)[x, y] \in J^2A$. \square

We remark that the same method of proof demonstrates that any $f \in K_{g,1}(n)$ satisfies $(f - 1)[x, y] \in J^n A$.

It appears to be a very difficult problem to compute the dimension of the space of coinvariants $A_{K_{g,1}}$. Since J^2A has finite codimension in A , this problem is equivalent to computing the codimension of $J_{K_{g,1}}A$ in J^2A . This problem can be solved “infinitesimally” in the sense that we can work out the precise relationship between these subspaces after passing to completions. The following is easily deduced from Corollary 6.

Proposition 8. *The inclusion $J_{K_{g,1}}A \subset J^{N_g}A$ induces an isomorphism*

$$(J_{K_{g,1}}A)^\wedge \cong (J^{N_g}A)^\wedge$$

on J -adic completions.

Question 1. Does $N_g = 2$?

Question 2. Is there an equality $J_{K_{g,1}}A = J^{N_g}A$?

An affirmative answer to Question 2 would imply that $A_{K_{g,1}}$ is finite-dimensional.

Proposition 9. *If $A_{K_{g,1}}$ is finite-dimensional, then $H_1(K_{g,1}, \mathbb{Q})$ is finite-dimensional as long as $g \geq 4$.*

5. AN EXTENSION OF A RESULT OF AKITA

By a direct generalization of the arguments of Akita, we will show that the higher Johnson subgroups must have infinite-dimensional homology in at least some degrees.

Lemma 5.1 ([1]). *Let $F \rightarrow E \rightarrow B$ be a fibration with B a finite CW complex. If $H_\bullet(F, \mathbb{Q})$ is finite-dimensional, then so is $H_\bullet(E, \mathbb{Q})$.*

Proof of Proposition 2. For each $n \geq 2$ there is a short exact sequence

$$1 \rightarrow K_{g,1}(n+1) \rightarrow K_{g,1}(n) \xrightarrow{\tau_{g,1}(n)} V_{\mathbb{Z}} \rightarrow 1$$

where $V_{\mathbb{Z}} \subset \text{Hom}(H, \mathcal{L}_n(\pi))$ is a free abelian group of finite rank. Thus for each n there is a fibration of classifying spaces of the form

$$BK_{g,1}(n+1) \rightarrow BK_{g,1}(n) \rightarrow \mathbb{T}_n$$

where \mathbb{T}_n is a compact torus of dimension $\text{rank}_{\mathbb{Z}}(V_{\mathbb{Z}})$. By Akita, it is known that $H_\bullet(K_{g,1}, \mathbb{Q})$ has infinite-dimension as long as $g \geq 7$. Appropriately modified statements hold with $K_g(n)$ in place of $K_{g,1}(n)$. An inductive argument combined with Lemma 5.1 finishes the proof. \square

We can say more in genus 2. Since $K_2 = T_2$ is a free group by [13], the subgroups $K_2(n)$ are all free when $n \geq 2$.

Proposition 10. *For each $n \geq 2$ there is a splitting $K_{2,1}(n) \cong K_2(n) \ltimes \pi^{(n)}$.*

Furthermore, since the cokernel of the inclusion $K_2(n+1) \rightarrow K_2(n)$ is a finitely generated group, we obtain the following by induction.

Proposition 11. *For each $n \geq 2$, the rational homology groups $H_1(K_{2,1}(n), \mathbb{Q})$ and $H_1(K_2(n), \mathbb{Q})$ are infinite-dimensional.*

Proof. It is clear from Mess's computation that $H_1(K_{2,1}, \mathbb{Q})$ is infinite-dimensional. The key observation, now, is that for each n the image of $H_1(K_{g,1}(n+1), \mathbb{Q})$ in $H_1(K_{g,1}(n), \mathbb{Q})$ has finite codimension. The result follows by induction. \square

It turns out that if the space of coinvariants $A_{K_{g,1}}$ is infinite-dimensional, then Proposition 2 can be considerably sharpened. If this condition were satisfied, then for all $n \geq 2$ at least one of $H_1(K_{g,1}(n), \mathbb{Q})$ and $H_2(K_g(n), \mathbb{Q})$ would be infinite-dimensional. We will prove this in two steps.

Lemma 5.2. *The space of coinvariants $A_{K_{g,1}}$ is infinite-dimensional if and only if the space of coinvariants $H_1(\pi^{(n)}, \mathbb{Q})_{K_{g,1}}$ is infinite-dimensional for all $n \geq 3$.*

Proof. Consider the exact sequence

$$H_1(\pi^{(n+1)}, \mathbb{Q})_{\pi^{(n)}} \rightarrow H_1(\pi^{(n)}, \mathbb{Q}) \rightarrow \mathcal{L}_n(\pi) \otimes \mathbb{Q} \rightarrow 0.$$

Because the operation of taking $K_{g,1}$ -coinvariants is right-exact, there is an exact sequence

$$H_1(\pi^{(n+1)}, \mathbb{Q})_{K_{g,1}} \rightarrow H_1(\pi^{(n)}, \mathbb{Q})_{K_{g,1}} \rightarrow \mathcal{L}_n(\pi) \otimes \mathbb{Q} \rightarrow 0,$$

as $K_{g,1}$ acts trivially on $\mathcal{L}_n(\pi)$. Since $\mathcal{L}_n(\pi) \otimes \mathbb{Q}$ is a finite-dimensional vector space, the result now follows by induction. \square

Proof of Theorem 1.2. Assume that $A_{K_{g,1}}$ is infinite-dimensional. Then by Lemma 5.2 so is $H_1(\pi^{(n)}, \mathbb{Q})_{K_{g,1}}$ for each $n \geq 2$. Since $K_{g,1}(n)$ is a subgroup of $K_{g,1}$, there is a surjection $H_1(\pi^{(n)}, \mathbb{Q})_{K_{g,1}(n)} \rightarrow H_1(\pi^{(n)}, \mathbb{Q})_{K_{g,1}}$. As the latter space is infinite-dimensional, this implies that $H_1(\pi^{(n)}, \mathbb{Q})_{K_{g,1}(n)}$ is as well. The 5-term exact sequence of the extension (2) has a segment

$$H_2(K_g(n), \mathbb{Q}) \rightarrow H_1(\pi^{(2)}, \mathbb{Q})_{K_{g,1}} \rightarrow H_1(K_{g,1}(n), \mathbb{Q}).$$

Therefore, at least one of $H_2(K_g(n), \mathbb{Q})$ and $H_1(K_{g,1}(n), \mathbb{Q})$ is infinite-dimensional. \square

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